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NOTE ON A TWISTED CURVE CONNECTED WITH AN INVOLUTION OF PAIRS OF POINTS IN A PLANE.*

BY H. S. WHITE.

1. A Method for Generating some Twisted Curves. The points of a plane are said to be in involution when they are so related in sets of n points that any one point of a set determines all the other $n-1$ points of that set. The simplest involutions, and the best known, are those where each set contains only 2 points. Of involutions of higher order so much is known as that they are all rational; that is, that between the points of a plane and the sets of points of any involution a one-to-one relation can be established.† But for orders higher than 2 the classification and reduction to types have not yet been given. Hence we shall understand for the present by "involution" always an involution of order 2, one whose points are grouped in pairs.

Bertini showed that involutions of second order are of four distinct types, every such involution falling into some one of these four classes.‡ Two of the same type are usually not projectively equivalent, but each can be transformed into the other by some Cremona transformation. As the representative of each type he chooses the simplest. The first type is that in which collineation or projection effects an exchange of the points of each pair. Such a collineation evidently must be a perspectivity, or central projection. The second type includes as many distinct species as there are possible orders of curves, one for every integer above 3. Among these we consider that of lowest order, in which paired points lie on rays passing through a fixed point, and are conjugate with respect to a fixed conic.§ These two alone are to be used in the present note.

The notion of an involution in a plane may be used to generate twisted curves, just as the notion of involution on a line has been used (and may be used further) to generate plane curves. To make this explicit, recall the method proposed by Steiner, and elaborated by Schroeter, for generating a cubic curve in the plane. A quadric involution on a straight line, it will be remembered, consists of pairs of points harmonic with respect to two fixed points, either real or imaginary, on the line. Two pairs of points determine fully such an involution, since there is only a single pair of points separating harmonically two given pairs. Any third pair of points, in order to belong to the same involution, must satisfy a single condition. Hence if three pairs of points are taken at random in a plane, and these are projected from a variable center in the plane upon any line, the requirement that their projections shall form 3 pairs in quadric involution will subject the variable center to a single condition; *i. e.* the locus of that point is a definite curve. In this case the locus is a general cubic curve.

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* Read before the Chicago Section of the American Mathematical Society at its meeting January 3, 1902.

† G. Castelnuovo, *Mathematische Annalen*, vol. 44 (1894), p. 125.

‡ Ricerche sulle trasformazioni univoche involutorie nel piano. *Annali di matematica*, ser. 2, vol. 8 (1877), p. 254.

§ Compare the system of intersections of the circles of a net, used as an illustration, by Professor Bôcher in the *Annals of Mathematics*, ser. 2, vol. 3 (1902), pp. 49–52, and especially the first footnote on p. 52.

In the same way, if a certain number of pairs of points in a plane suffice to determine an involution of a specified type, then one additional pair of points, in order to belong in the same involution, must satisfy two conditions. Assume now, in space of 3 dimensions, pairs of points one more than sufficient; project them upon a plane from a variable center, and require the projections to form pairs in an involution of the specified type. This will be equivalent to 2 conditions, restricting the variable center to motion along some definite twisted curve.

In this way involutions of order 2 can give rise to an infinite variety of algebraic twisted curves, and may be found to lead to important properties of those curves. Indeed two species of involution in the plane which are equivalent to each other by virtue of some Cremona transformation will usually be connected with twisted curves which are not transformable into one another by any Cremona transformation of three dimensional space.

To exhibit the method I have worked out the following two simplest examples.

2. First Example: the Perspective Involution. In a perspective relation of the points in a plane, the axis o and the center O may be any line and any point. To find the conjugate to any point P_1 (see Fig. 1), produce a line OP_1 to cut the axis o in a point P', and determine a fourth point P_2 on OP_1 harmonic to O, P_1, and P'. All the pairs of points P_1P_2 constitute then an involution of the first type. Otherwise, two such pairs may be taken arbitrarily, and from them the center, the axis, and consequently all other pairs can be found. Suppose A_1 and A_2, B_1 and B_2 to denote given pairs, then any third pair $P_1 P_2$ must have their joining line pass through the point O where the line $A_1 A_2$ meets $B_1 B_2$. This is the first condition. Again, this center O determines two points A' and B' where the axis o must intersect $A_1 A_2$ and $B_1 B_2$; the second condition is that the center O and axis o shall separate harmonically the points P_1 and P_2.

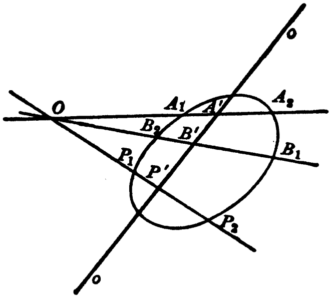

FIG. 1.

Assume in space three pairs of points $A_1 A_2$, $B_1 B_2$, $C_1 C_2$ (no four in any plane). If the projections of the 3 lines a, b, c which join these several pairs are to meet in one point of a plane, the center of projection X must lie on a line intersecting all three: a, b, and c. That is, X must lie on a ruled quadric having a, b, c as directrices. This first condition restricts X to a quadric surface.

The second condition is satisfied if the six projecting rays XA_1, XA_2, XB_1, . . . XC_2, lie on a quadric cone. For then the generator which passes through X, and its polar plane with respect to the cone, will project from X into the required center and axis of perspectivity, inasmuch as their sections by any plane will separate harmonically the projections of each pair of given points, *e. g.* of A_1 and A_2. Conversely we see that this condition is not only sufficient, but also necessary for the point X. We have to find therefore the locus of the vertex of a cone passing through 6 given points. That this is a surface of the fourth order* can be shown as follows.

A quadric surface is fixed by 9 points. Hence through the 6 given points there pass all the quadrics of a linear system with 3 arbitrary parameters:

$$\phi_0 + \lambda_1 \phi_1 + \lambda_2 \phi_2 + \lambda_3 \phi_3 = 0.$$

If X is to be a conical point upon one of these quadrics, 4 conditions must be satisfied, *viz.* the vanishing of the 4 partial derivatives:†

$$\phi_{01}(x) + \lambda_1 \phi_{11}(x) + \lambda_2 \phi_{21}(x) + \lambda_3 \phi_{31}(x) = 0,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\phi_{04}(x) + \lambda_1 \phi_{14}(x) + \lambda_2 \phi_{24}(x) + \lambda_3 \phi_{34}(x) = 0.$$

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* The "Surface of Weddle." See full discussion in Cayley's *Collected Works*, vol. 7, p. 160 sqq.; and a paper by Chasles, *Comptes Rendus*, vol. 52 (1861) pp. 1157–1162.

† The quadrics ϕ_0, ϕ_1, etc., are supposed to be homogeneous in the coordinates x_1, x_2, x_3, x_4. Partial derivatives are then denoted by added subscripts. Thus the abbreviation $\phi_{21}(x)$ means $\frac{\partial}{\partial x_1} \phi_2 (x_1 x_2, x_3, x_4)$.

Eliminating from these 4 equations, linear in the (x), the parameters $\lambda_1, \lambda_2, \lambda_3$, we find the equation of a quartic locus for X. Notice that it is the Jacobian of all quadric surfaces that pass through the 6 fixed points.

The intersection of a quadric and a quartic is in general an octavic curve. But this quartic contains entire lines, among them the 15 which join two and two the 6 fixed points; and 3 of these are the directrices a, b, c of the quadric. Hence the proper curve is a quintic, and cuts every directrix in 4 points, but every generator of the quadric (*i. e.* those of the system opposite to a, b, c) in only 1 point. This quintic belongs obviously to the class a_0' of Noether,[*] a sub-species not particularly described by Halphen,[†] but included under his second species of quintics. All directrices of the quadric surface are four-fold secants of the curve, so that its projection from any point of the curve upon a plane is a plane quartic with a triple point, proving that the quintic is of deficiency zero, *i. e.* it is rational. Generators of the second system meet it in only one point each, hence it can have no actual double point. Finally, a counting of constants makes it probable that every quintic of Noether's class a_0' can be generated by the above method; if this is true, a geometric proof would disclose an interesting property of the curve.

3. Second Example: Involution of Harmonic Conjugates on Rays Cutting a Conic. Five pairs of conjugate points with respect to a conic completely determine that conic. Hence 5 pairs of points suffice to determine an involution of the second type. But these 5 pairs are not perfectly arbitrary; they are subject to the restriction that the 5 lines joining the respective pairs must have a common point, which we may call a *radiant point*. Any sixth pair of points, in order to belong in the same involution as the first five, must satisfy 2 conditions: first, their joining line must pass through the radiant point; and second, the two points must be conjugate with respect to the same conic which separates harmonically the first 5 pairs. In Fig. 2 O is the radiant point, $A_1 A_2$, $B_1 B_2$, etc., the 5 pairs that determine the involution, o the conic which separates each pair harmonically, and $P_1 P_2$ any sixth pair in the involution.

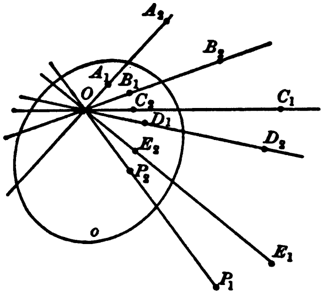

FIG. 2.

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[*] Zur Grundlegung der Theorie d. algebraischen Raumcurven. *Abhndlgn. der kgl. Akad. d. Wissenschaften zu Berlin*, 1882, Anhang, p. 83.

[†] Sur la classification des courbes gauches algébriques. *Jour. de l'Ecole polytechnique*, vol. 52 (1882), p. 162.

To secure a variable center of projection we have therefore to assume in space, not 6 pairs of points taken at random, but one pair on each of 6 directrices of a ruled quadric surface. Then a center of projection X anywhere upon that quadric will give a set of projections satisfying the first condition, for the 6 directrices will project into lines meeting upon the generator that passes through X. As to the second condition, some quadric surface separating harmonically each of the 6 given pairs of points must be a cone having X for its vertex.* Each pair of points, as being conjugates, supplies one linear condition for the coefficients of the equation of that quadric, leaving 3 free parameters $(10 - 6 - 1 = 3)$. Here therefore the reasoning of §2 applies again, and the locus of the vertex X is a quartic surface, the Jacobian of 4 quadrics that have generally no common point. This Jacobian must evidently contain each one of the 12 given points, but need not contain any of their join lines. The curve of intersection is accordingly of order 8 and is of the species a_9 of Noether (*l. c.*, p. 96), or Halphen's 1st species (*l. c.*, p. 165).

By counting constants we may judge that every curve of this species lies on at least one Jacobian surface, and contains 24 pairs of points separated harmonically by all cones of the system. But that these pairs, or any six of them, will lie on as many of the quarti-secants of the curve is apparently improbable. Hence the curve found from six arbitrary pairs of points on six generators of a quadric by the aid of the concept, involution of the second type, must be of a very special kind under its species. To confirm this, it would be needful to study minutely the system of lines joining corresponding points on the general Jacobian surface.

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* See Cayley and Chasles, as cited above.